On blowup in semilinear wave equations

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Outline

- Nonlinear Genera Relativity equations - Black Holes.
- Nonlinear Navier-Stokes equations - turbulence, formation of dorps,...

Introduction

- Nonlinear Genera Relativity equations - Black Holes.
- Nonlinear Navier-Stokes equations - turbulence, formation of dorps,...

Base equation

$$U_{tt} - \triangle U + \mu U - U^p = 0, \qquad U = U(x,t), \qquad x \in R^n. \qquad (1)$$

- $\mu = 0$ NLW
- $\mu = 1$ NLKG

Scaling

- Used by enginieers by building small-scale mockups that resemble oryignal-scale model.
- Dimension analysis of nuclear explosion by G.I. Taylor.
- Supernova blast.

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Symmetries

Equation symmetry

$$U_\lambda(t,r) = \lambda^\alpha U\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right), \qquad \alpha = \frac{2}{p-1}.$$

Energy functional

$$E[U] = \int_{\mathbf{R}^n} \left(\frac{1}{2}U_t^2 + \frac{1}{2}(\nabla U)^2 - \frac{1}{p+1}U^{p+1}\right) d^n x,$$

scales as

$$E[U_\lambda] = \lambda^\beta E[U],$$

where $\beta = \frac{(n-2)p-(n+2)}{p-1}$.

Scaling is:

- critical for $\beta = 0$, i.e. $p = p_Q = \frac{n+2}{n-2}$,
- subcritical $p < p_Q$,
- supercritical $p > p_Q$.

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Self-similar solutions

Definition

$$U(t,r) = (T-t)^{-\alpha} u(\rho), \qquad \rho = \frac{r}{T-t} \tag{2}$$

Equation for self-similar profiles

$$(1-\rho^2)u'' + \left(\frac{n-1}{\rho} - \frac{2(p+1)}{p-1}\rho\right)u' - \frac{2(p+1)}{(p-1)^2}u + u^p = 0 \tag{3}$$

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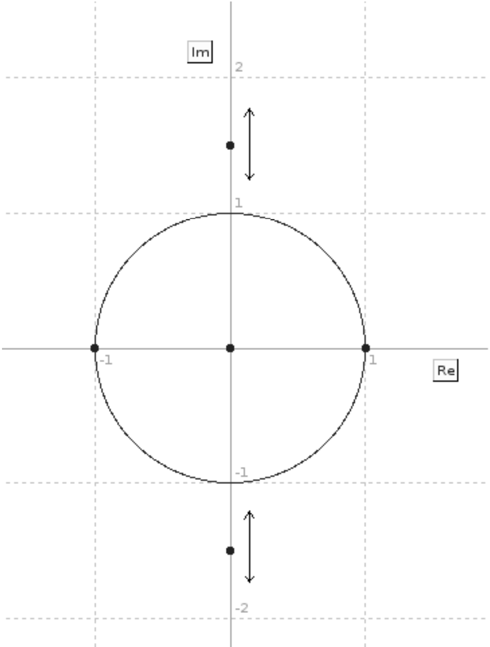
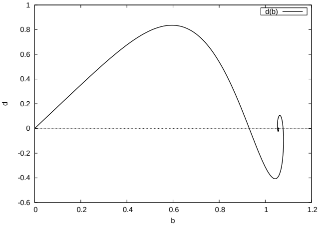
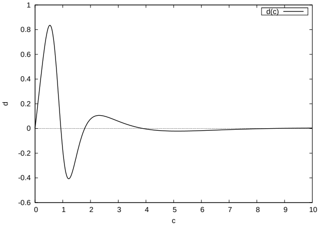
Self-similar solutions



Self-similar solutions

- Solution around $\rho = 0$ are simple Taylor series.
- Solutions around $\rho = 1$ are:
  - simple Taylor series for $k = \frac{(n-1)p-n-3}{2(p-1)}$ - noninteger;
  - complicated when k - integer (resonance condition);
- These solutions can be matched - we get global solution on $\rho \in [0; 1]$.Countable family of self-similar profiles.
- There is simple numerical method to construct these solutions.





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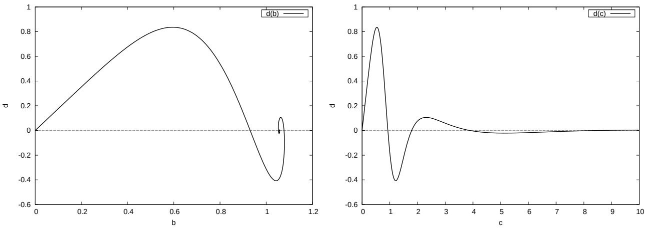

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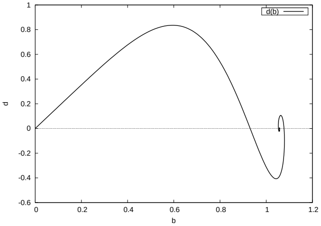

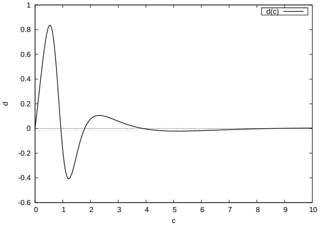

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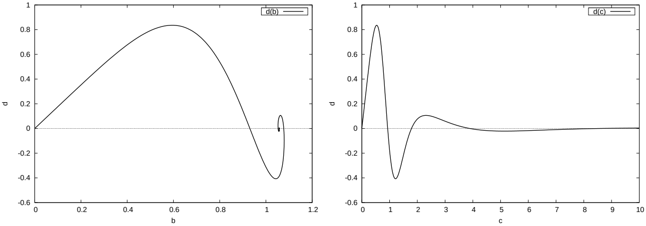

Self-similar solutions

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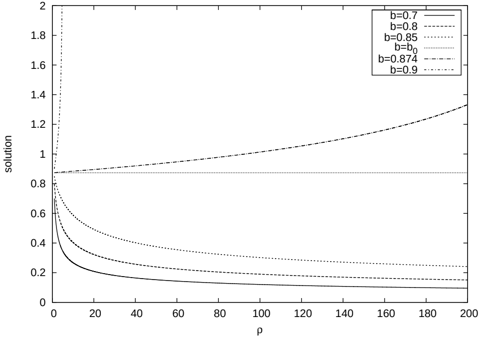
- Global solution vanish when $\rho \to \infty$.



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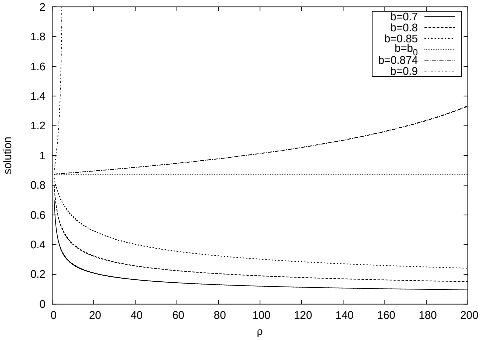


How similar the evolution of (NLW)

$$U_{tt} - U_{rr} - \frac{n-1}{r} U_r - U^p = 0 \tag{4}$$

is to the evolution of (NLKG)

$$U_{tt} - U_{rr} - \frac{n-1}{r} U_r + U - U^p = 0. \tag{5}$$

Conjecture

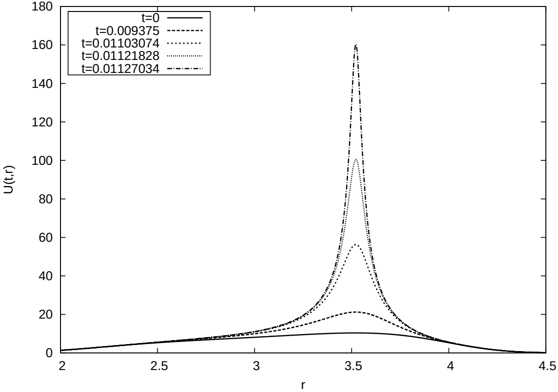
Blowup asyptotics of (4) is structurally stable under the perturbation resultin in adding the mass term.

ODE blowup

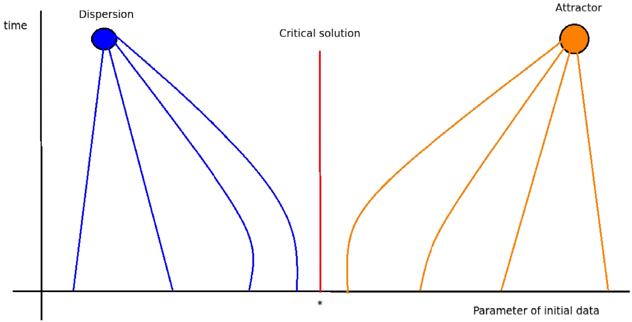
Time independent solution for NLW:

$$U_0(t) = \frac{b_0}{(T-t)^\alpha}, \qquad b_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}$$

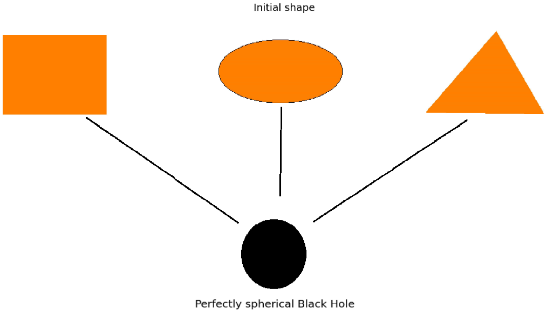
is stable and is the attractor for the solutions that blows up in finite time. It is true for arbitrary spatial dimension.



Intermediate attractors



Intermediate attractors - no hair theorems BH

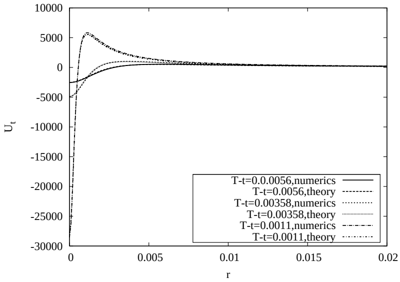


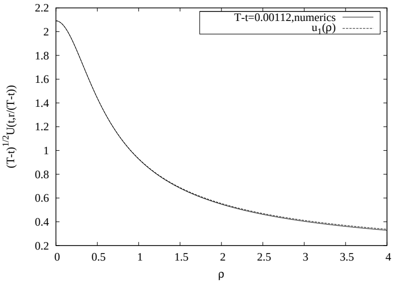
Supercritical cases

- l-th self-similar solution has l unstable modes.
- First solution is intermediate attractor (even for special k cases):

$$U(t,r) = \frac{u_1(\rho)}{(T-t)^{\alpha}} + C\frac{\xi_1(\rho)}{(T-t)^{\lambda_1+\alpha}} + \text{damped} \quad \text{modes},$$

- For NLKG intermediate asymptotics „converges“ to the first self-similar solution of NLW.





Similar plots can be obtained for NLKG.

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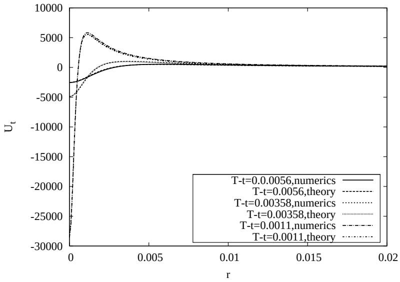

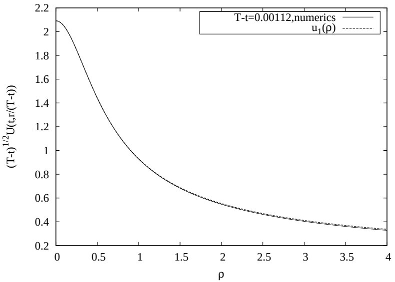

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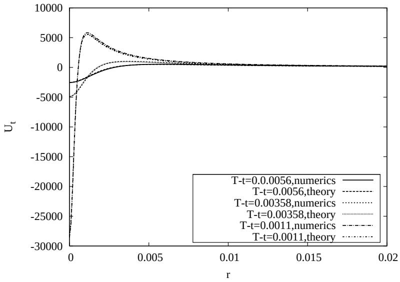

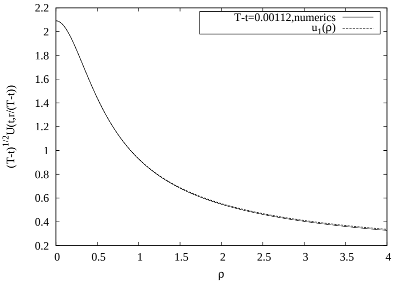

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Critical cases

- There exist only one self-similar solution (apart from the trivial one).
- Static solution plays special role (it is unstable - one mode):

$$Q(r) = \frac{1}{(1+br^2)^\alpha}, \quad b = \frac{p-1}{4n}, \quad Q_\lambda(r) = \frac{1}{\lambda^\alpha} Q(r/\lambda). \quad (6)$$

- Possible behaviours:
  - Type I blowup - generic one.
  - Type II blowup - (6) z $lim_{t \to T} \lambda(t) \to 0$.
  - Solution converges to static solution.
  - „Dispersion"

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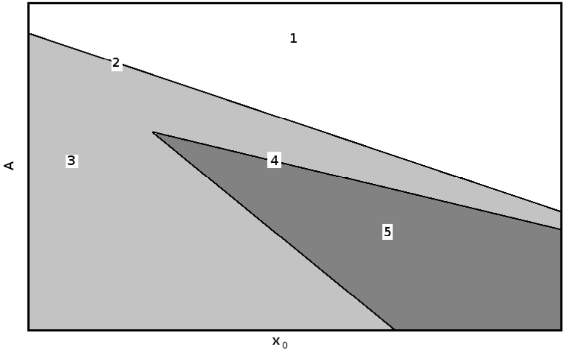
Critical cases

Numerical simulations with initial conditions:

$$U(t=0,r) = Ar^2 e^{-\left(\frac{r-x_0}{s}\right)^4}, \qquad U_t(t=0,r) = 0;$$

A and x_0 are parameters, s fixed.



NLKG - $n = 3$, $p = 3$

$$U_{tt} - U_{rr} - \frac{2}{r} U_r + U - U^3 = 0,$$

- Static solution is unstable attractor

$$S_{rr} + \frac{2}{r} S_r - S + S^3 = 0$$

with $S(0) = 4.34...$.

- Expansion around the attractor

$$U(t, r) \approx S(r) + A_0 e^{s_0 t} v_0(r) + C \frac{\sin(t + \delta)}{t^{3/2}} v(\lambda = 0, r).$$

- Numeric suggests damping of the type $\sim t^{-1/2}$.

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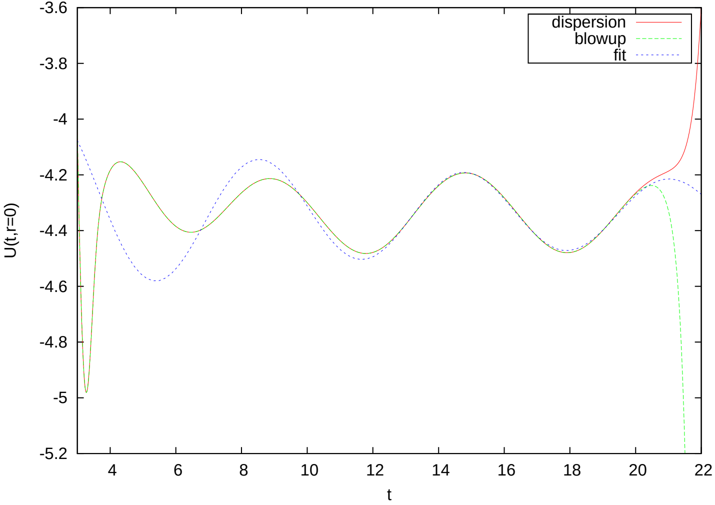

dispersion
blowup
fit
U(t,r=0)
t
-3.6
-3.8
-4
-4.2
-4.4
-4.6
-4.8
-5
-5.2
4
6
8
10
12
14
16
18
20
22

A few strict results...
...what mathematicans can say about this.

(some results of J. Krieger, K. Nakanishi, W. Schlag, O. Costin, D. Tataru, R. Donninger, P. Raphael,...)

Singularity formation for (some) nonlinear equations [2]

$$P \frac{d^2 u}{dt^2} = -A(t)u + F(u), \quad t \in [0, T), \quad u(0) = u_0, \quad u_t(0) = v_0$$

- $u(t)$ mapping form some region to Hilbert space;
- $A(t)$ - symmetric, non-negative operator for $t \geq 0$ (e.g. Laplace operator);
- P - symmetric, positive operator (e.g. identity);
- F - nonlinear term (e.g. polynomial);

Singularity formation for (some) nonlinear equations [2]

Let F has „potential functional” given by

$$V(x) = \int_0^1 (F(tx), x)dt,$$

and let exists $\alpha > 0$, that

$$(x, F(x)) \geq 2(2\alpha + 1)V(x),$$

then, if

$$V(u_0) > \frac{1}{2}[(u_0, A(0)u_0) + (v_0, Pv_0)] =: E_{lin}(0),$$

then u exists only for finite times $T < \infty$:

$$lim_{t\to T^-}(u, Pu) = +\infty.$$

When $V(u_0) < E_{lin}(0)$ could have exists for arbitrary times...

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Types of behaviour for critical $p = 5$

For radial solution u in critical case there is possible to develop into:

- **Type I blowup**: $T_+(u) < \infty$ and

$$lim_{t \to T_+} ||(u(t), \partial_t u(t))||_{\dot{H}^1 \times L^2} = \infty$$

- **Type II blowup**: $T_+(u) < \infty$ and there exist the functions $(v_0, \partial_t v_1) \in \dot{H}^1 \times L^2$, the number $J \in \mathbf{N} \setminus \{0\}$, that for $j \in \{1, \ldots, J\}$, and for signs of $i_j \in \{\pm\}$ and for the positive functions $\lambda_j(t)$ defined for t close to T_+ we have

$$\lambda_1(t) \ll \ldots \ll \lambda_J(t) \ll T_+ - t \quad \text{dla} \quad t \to T_+,$$

$$lim_{t \to T_+} \left\| (u(t), \partial_t u(t)) - \left(v_0 + \sum_{j=1}^{J} \frac{i_j}{\lambda_j(t)^{1/2}} Q\left(\frac{x}{\lambda_j(t)}\right), v_1 \right) \right\|_{\dot{H}^1 \times L^2} = 0$$

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Types of behaviour for critical $p = 5$

- **Global solution**: $T_+(u) = \infty$ and there exist the solution v_L of the **linear** equation, the number $J \in \mathbf{N}$, that for $j \in \{1, \ldots, J\}$, and for the sign of $i_j \in \{\pm\}$ and positive functions $\lambda_j(t)$ defined for large t we have

$$\lambda_1(t) \ll \ldots \ll \lambda_J(t) \ll t \quad \text{dla} \quad t \to \infty,$$

$$lim_{t \to +\infty} \left\| (u(t), \partial_t u(t)) - \left(v_L(t) + \sum_{j=1}^{J} \frac{i_j}{\lambda_j(t)^{1/2}} Q\left(\frac{x}{\lambda_j(t)}\right), \partial_t v_L(t) \right) \right\|_{\dot{H}^1 \times L^2} = 0,$$

***

$f(t) \ll g(t)$ for $t \to L$ when $lim_{t \to L} \frac{f(t)}{g(t)} = 0.$

„Energy" for critical case $p = 5$

- If $E(u) < 0$ then solution blows up in finite time

$$lim_{t \to T_+}(||u(x,t)||_{H^1} + ||\partial_t u(x,t)||_{L^2}) = \infty;$$

- If $E(u_0, v_0) < E(Q, 0)$ then:
  - for $|| \bigtriangledown u_0|| < || \bigtriangledown Q||$ - the solutions exists for all times;
  - for $|| \bigtriangledown u_0|| > || \bigtriangledown Q||$ - the solutions blows up.;
- If $E(u_0, v_0) = E(Q, 0)$ (energy threshold) then for data $|| \bigtriangledown u_0|| + ||v_0|| \leq || \bigtriangledown Q||$ we have global existence; the only solutions with this class that do not disperse are $\pm Q$.

„Energy" for critical case $p = 5$

- **Blowup in finite time (type I + II)**:

$$lim_{t \to T_+} ||(u(t), \partial_t u(t))||_{\dot{H}^1 \times L^2} \in (|| \bigtriangledown Q||, \infty]$$

- **Global existence**:

$$lim_{t \to T_+} ||(u(t), \partial_t u(t))||_{\dot{H}^1 \times L^2} = A,$$

such that

$$2E(u_0, u_1) \leq A \leq 3E(u_0, u_1).$$

(u_0, u_1) - initial data with $\dot{H}^1 \times L^2$.

Subcritical NLKG $n = 3$, $p = 3$

Let S be a radial solution of static equation

$$-\triangle S + S = S^3.$$

The energy of this solution fulfills

$$E(S) =: J(S) > 0.$$

Solutions below the energy of static solution $E(u) < E(S)$ depending on the value of

$$K_0(u) := \partial_\lambda J(\lambda u)|_{\lambda=1}$$

develop into

- $K_0(u(0)) \leq 0$ - global solutions for all times;
- $K_0(u(0)) > 0$ - blowup in finite time $\pm T$;

There also exist „one pass theorems“ that say that the solution can reach the neighborhood of $\pm S$ only once during the evolution !

Selected bibliography

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